SISBayes - Padova

Multivariate Causal Effect: a Bayesian Regression Factor Model

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joint work with
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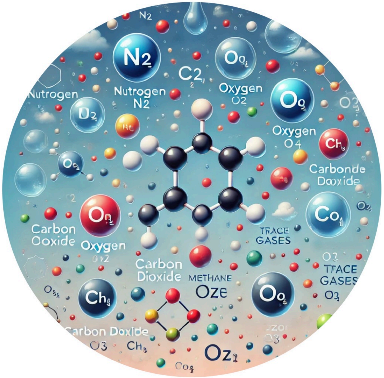
Data Science Institute

- Epidemiological studies demonstrate that exposure to higher levels of $\mathrm{PM}_{2.5}$ corresponds to higher mortality/diseases risk.



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- The fine particular matter $\mathrm{PM}_{2.5}$ is composed by different chemical components.
- Some events, as wildfires, can change the chemical compositions.

Studying causal effects of wildfire smoke on $PM_{2.5}$ chemicals

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Multivariate Outcome

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Factor Models

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Causality

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Potential Outcome Framework

Outline

1. Potential Outcome Framework
2. Bayesian Causal Factor Model
3. Simulation Study
4. Environmental Application

Potential Outcome Framework

Rubin (1974)'s Potential Outcome Framework

$T_i \in \{0, 1\}$

T

Treatment

Rubin (1974)'s Potential Outcome Framework

$T_i \in \{0, 1\}$

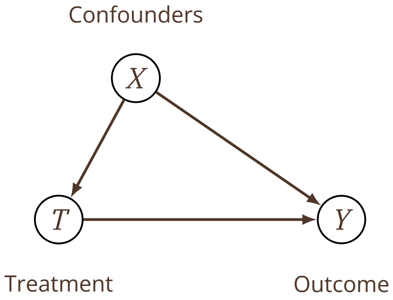
$$\begin{bmatrix} Y_{i1} \\ \dots \\ Y_{iq} \end{bmatrix} = \begin{cases} \begin{bmatrix} Y_{i1}(0) \\ \dots \\ Y_{iq}(0) \end{bmatrix} & \textit{if } T_i = 0 \\ \begin{bmatrix} Y_{i1}(1) \\ \dots \\ Y_{iq}(1) \end{bmatrix} & \textit{if } T_i = 1 \end{cases}$$

$$T \longrightarrow Y$$

Treatment

Outcome

Rubin (1974)'s Potential Outcome Framework



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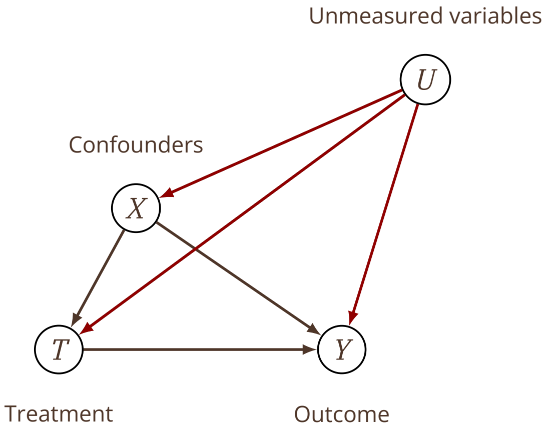

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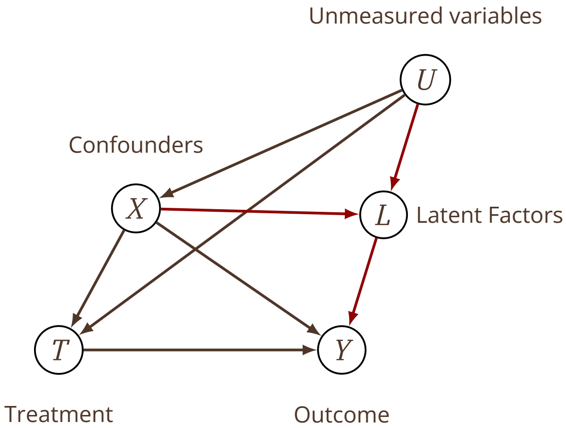

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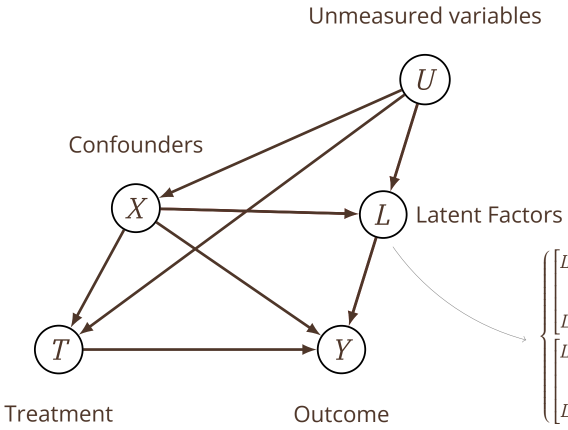

$$\begin{cases} \begin{bmatrix} L_{0,i1} \\ \dots \\ L_{0,iq} \end{bmatrix} & \textit{if } T_i = 0 \\ \begin{bmatrix} L_{1,i1} \\ \dots \\ L_{1,iq} \end{bmatrix} & \textit{if } T_i = 1 \end{cases}$$

Average treatment effects: $\begin{bmatrix} \tau_1^P \\ \dots \\ \tau_q^P \end{bmatrix} = \mathbb{E}\left(\begin{bmatrix} Y_{i1}(1) \\ \dots \\ Y_{iq}(1) \end{bmatrix} - \begin{bmatrix} Y_{i1}(0) \\ \dots \\ Y_{iq}(0) \end{bmatrix} \right)$

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$$\mathbf{Y}_i = (1 - T_i)\,\mathbf{Y}_i(0) + T_i\,\mathbf{Y}_i(1)$$

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3. Conditional ignorability:

$$\{\mathbf{Y}_i(1), \mathbf{Y}_i(0)\} \perp T_i \mid \{\mathbf{X}_i = \mathbf{x}_i, \mathbf{L}_{i0} = \mathbf{l}_{i0}, \mathbf{L}_{i1} = \mathbf{l}_{i1}\}$$

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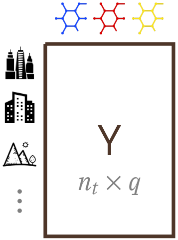
4. Indirectness of unmeasured confounding effects:
$$\mathbb{P}(\mathbf{Y}(1), \mathbf{Y}(0) \mid \mathbf{X}, \mathbf{L}_0, \mathbf{L}_1, U) = \mathbb{P}(\mathbf{Y}(1), \mathbf{Y}(0) \mid \mathbf{X}, \mathbf{L}_0, \mathbf{L}_1)$$

Bayesian Causal Factor Model

Bayesian Causal Regression Factor Model



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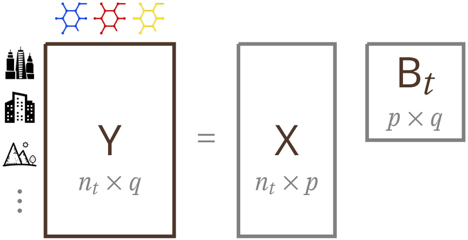

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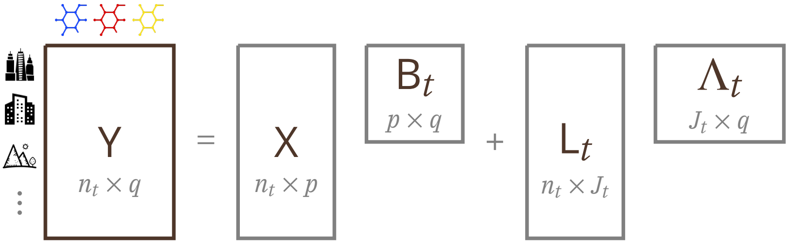

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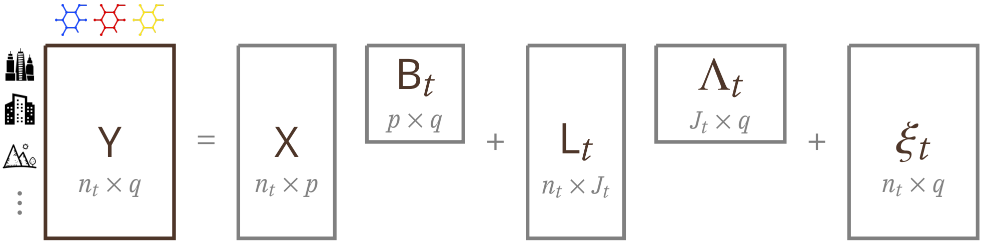

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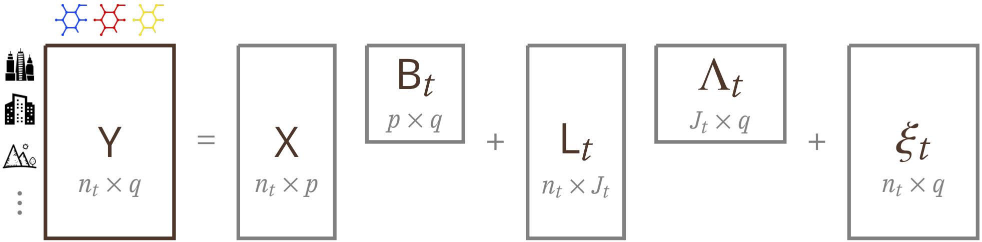

$$\{\mathbf{Y}_i \mid \mathbf{x}_i, t, \mathbf{l}_{it}\} = \boldsymbol{\mu}_t + \mathbf{B}_t\mathbf{x}_i + \Lambda_t\mathbf{l}_{it} + \boldsymbol{\xi}_{it}$$

$$\underset{n_t \times q}{\mathrm{Y}} = \underset{n_t \times p}{\mathrm{X}}\,\underset{p \times q}{\mathrm{B}_t} + \underset{n_t \times J_t}{\mathrm{L}_t}\,\underset{J_t \times q}{\Lambda_t} + \underset{n_t \times q}{\xi_t}$$

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Observed variables

- multivariate outcomes: Y
- covariates: X

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Observed variables

- multivariate outcomes: Y
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Regression coefficients

$$\beta_{jt} \sim \mathcal{N}_q(\mu_\beta, \Sigma_\beta)$$
$$\text{for } i \in \{1, \ldots, n\} \; t \in \{0, 1\}$$

$$\underset{n_t \times q}{\mathrm{Y}} = \underset{n_t \times p}{\mathrm{X}}\ \underset{p \times q}{\mathrm{B}_t} + \underset{n \times J_t}{\mathrm{L}_t}\ \underset{J_t \times q}{\Lambda_t} + \underset{n_t \times q}{\xi_t}$$

Treatment-specific factor scores

Dependent Dirichlet Process:

$$l_{itj} \mid \mathbf{x}_i, \mu, \sigma^2 \sim \sum_{r \geq 1} \pi_{tjr}(\mathbf{x}_i) \mathcal{N}(\mu_r, \sigma_r^2)$$

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Treatment-specific factor scores

Dependent Dirichlet Process: probit stick-breaking process (Rodriguez and Dunson, 2011)

$$l_{itj} \mid \mathbf{x}_i, \mu, \sigma^2 \sim \sum_{r \geq 1} \pi_{tjr}(\mathbf{x}_i) \mathcal{N}(\mu_r, \sigma_r^2)$$

$$\pi_{tjr}(\mathbf{x}_i) = V_{tjr}(\mathbf{x}_i) \prod_{g<r} \{1 - V_{tjg}(\mathbf{x}_i)\}$$

$$V_{tjr}(x_i) = \Phi(a_{tjr}(x_i)) \quad a_{tjr}(x_i) \sim \mathcal{N}(x_i^T \alpha_{tjr}, 1)$$

$$\underset{n_t \times q}{\mathrm{Y}} = \underset{n_t \times p}{\mathrm{X}}\;\underset{p \times q}{\mathrm{B}_t} + \underset{n_t \times J_t}{\mathrm{L}_t}\;\underset{J_t \times q}{\Lambda_t} + \underset{n_t \times q}{\xi_t}$$

Treatment-specific factor loadings

Multiplicative gamma shrinkage prior (Bhattacharya and Dunson, 2011)

$$\lambda_{tjh} \mid \kappa_{tjh}, \iota_{th} \sim N\left(0, \kappa_{tjh}^{-1} \iota_{th}^{-1}\right)$$

$$\forall j = \{1 \ldots J_t\} \quad t = \{0, 1\} \quad k = 1 \ldots \infty$$

$$\kappa_{tjh} \sim Ga(\nu_t/2, \nu_t/2)$$

$$\iota_h = \prod_{l=1}^{h} \delta_{tl} \quad \delta_{t1} \sim Ga\left(a_{t1}, 1\right) \quad \delta_{tl} \sim Ga\left(a_{t2}, 1\right) \quad l \geq 2$$

$$\underset{n_t \times q}{\mathrm{Y}} = \underset{n_t \times p}{\mathrm{X}}\,\underset{p \times q}{\mathrm{B}_t} + \underset{n_t \times J_t}{\mathrm{L}_t}\,\underset{J_t \times q}{\Lambda_t} + \underset{n_t \times q}{\xi_t}$$

Error matrix

$$\xi_{it} \sim \mathcal{N}_p(0, \mathrm{diag}(\psi_{i1t}, \ldots, \psi_{ipt}))$$
$$\text{for } i \in \{1, \ldots, n\} \; t \in \{0, 1\}$$

Simulation Study

Table 1

Parameters and distributions for the simulation scenarios.

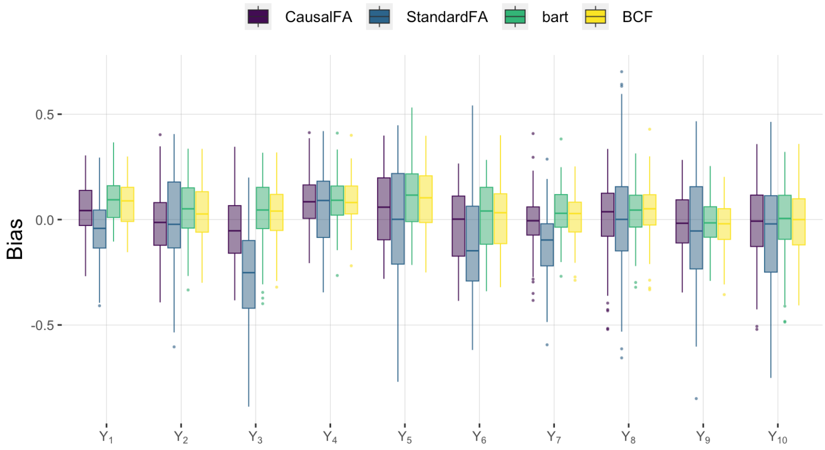
	Scenario 1	Scenario 2	Scenario 3	Scenario 4
Dimensions:				
n	500	500	500	3426
J_t	$(3,3)$	$(3,3)$	$(3,3)$	$(3,3)$
q	10	10	10	27
p	4	4	4	28
Variables:				
U	$U \sim N(0,2)$	$U \sim N(f_U(X_{1:4}), 0.5)$	$U \sim N(0,2)$	-
$\mathbf{X}$	$X_{1:2} \sim N(0,1)$	$X_{1:2} \sim N(0,1)$	$X_k \sim N(f_k(U), 1)$, $k=1,2$	observed $\mathbf{X}$
	$X_k \sim Be(\pi_k)$, $k=3,4$	$X_k \sim Be(\pi_k)$, $k=3,4$	$X_k \sim Be(\pi_k)$, $k=3,4$	
T	$T \sim Be(f_T(X_{1:4}, U))$	$T \sim Be(f_T(X_{1:4}, U))$	$T \sim Be(f_T(X_{1:4}, U))$	observed T
Factors:				
Clusters	3 or 2 cluster for each treatment-specific factors: $C_{ith} := f_C(\mathbf{X}_{1:4})$			$C_{ith} := f_C(\mathbf{X}_{1:27})$
l_{it}	$\{l_{ith} \mid C_{ith} = c\} \sim N(\mu_{tc} + \gamma_{tc}U, 1),\ h = \{1, \dots, p\}$			$\{l_{itj} \mid C_{ith} = c\} \sim N(\mu_{tc}, 1)$
Λ_t	matrices with 25% of zeros and nonzeros elements generated			estimated Λ_t
	by $\lambda_{th} \sim \pi Unif[-1, -0.8] + (1-\pi) Unif[0.8, 1]$, with $\pi \sim Be(0.5)$			in observed data
Outcome mode:				
B_t	$\beta_{ht} \sim Unif[-3+t, 2+t],\ t = 0,1,\ h = 1, \dots, p$			
ϵ_t	$\xi_{it} \sim Unif_p[0,1],\ t = 0,1$			
$\mathbf{Y}$	$Y_i(t) = B_t X + \Lambda_t l_{it} + \xi_{it},\ t = 0,1\ i = 1, \dots, n$			

Simulation study

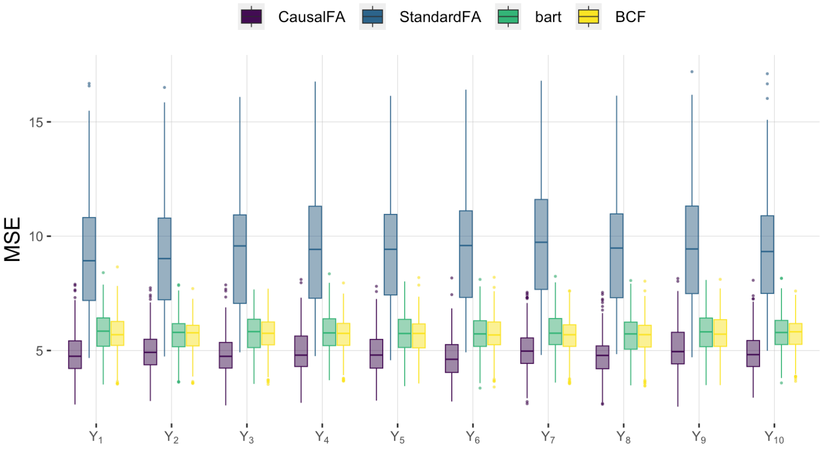
Models comparison:

- **CausalFA**: our regression factor model with mixture distribution for the factor score;
- **StandardFA**: regression factor model with standard prior: $l_t \sim \mathcal{N}(0, \mathbb{I}_{J_t})$;
- **BART** (Hill, 2011): estimated for each component of the multivariate outcome independent;
- **BCF** (Hahn, et al., 2020): estimated for each component of the multivariate outcome independent.

Comparison results: bias

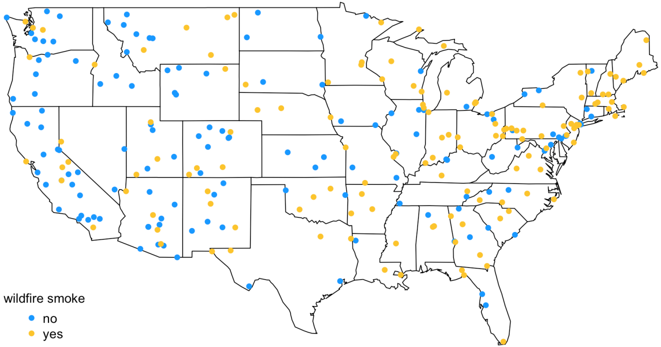


Comparison results: MSE



Environmental Application

Causal effect of wildfire smoke on ambient chemical



Dataset sources: Krasovich Southworth et al. (2025), Federal Land Manager Environmental Database (F.E.D. IMPROVE, 2024), GitHub repository echolab-stanford/daily-10km-smokePM (Child, et al 2022), Harvard Dataverse (Audirac, 2024), tidycensus R package (Walker et al., 2021).

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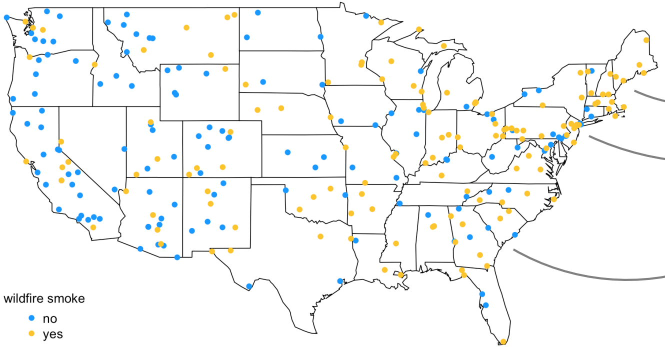

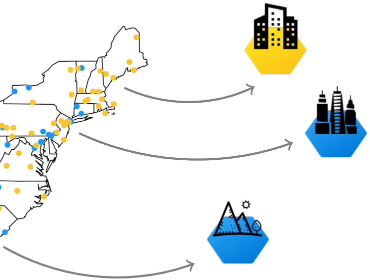

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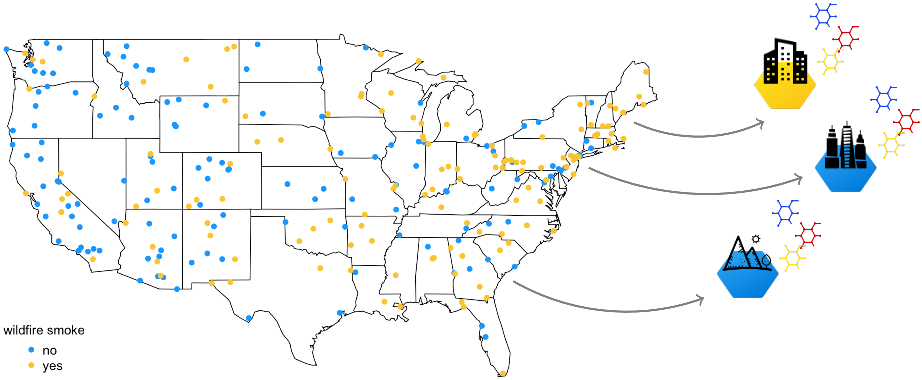

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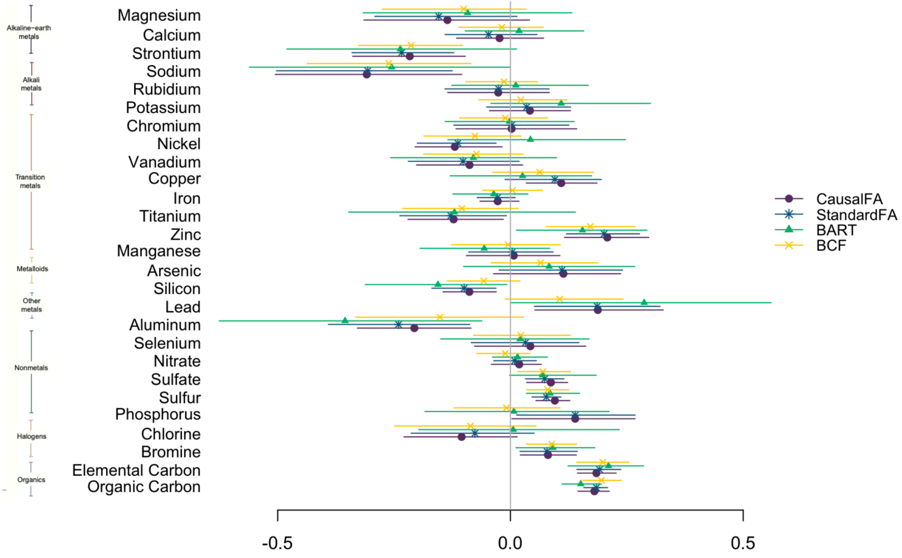
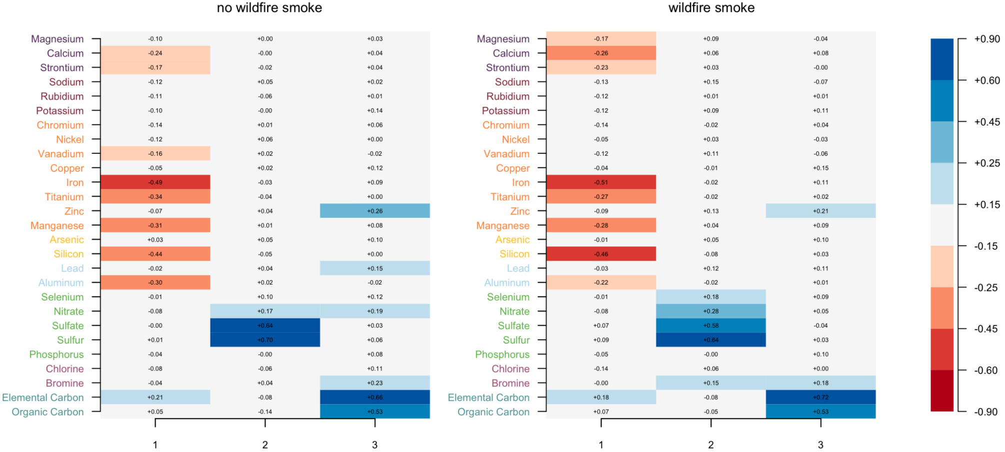


Treatment-specific factor loadings



Wrapping up

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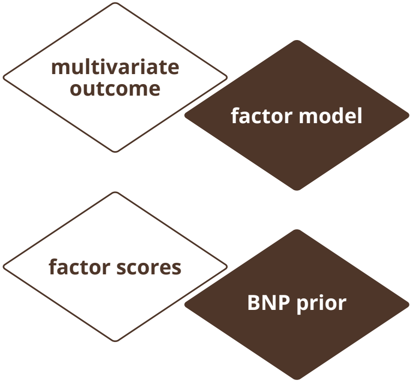

- We define a causal inference framework for multivariate potential outcomes.
- We introduce a Bayesian regression factor model for causal inference framework.
- We employs a dependent Dirichlet process as distribution for treatment-specific factor scores for overcome missing data problem.

Thanks for your attention

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